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Possibilistic networks: computational analysis of *MAP* and *MPE* inference

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Possibilistic graphical models are powerful modeling and reasoning tools for uncertain information based on possibility theory. In this paper, we provide an analysis of computational complexity of *MAP* and *MPE* queries for possibilistic networks. *MAP* queries stand for maximum a posteriori explanation while *MPE* ones stand for most plausible explanation. We show that the decision problems of answering *MAP* and *MPE* queries in both min-based and product-based possibilistic networks is *NP*-complete. Definitely, such results represent an advantage of possibilistic graphical models over probabilistic ones since *MAP* queries are NP^{PP}-complete in Bayesian networks. Our proofs for querying min-based possibilistic networks make use of reductions from the 3SAT problem to querying possibilistic networks decision problem. Moreover, the provided reductions may be useful for the implementation of *MAP* and *MPE* inference engines based on the satisfiability problem solvers. As for product-based networks, the provided proofs are incremental and make use of reductions from SAT and its weighted variant WMAXSAT.

Keywords: Complexity, Possibilistic networks, *MAP* inference, *MPE* inference

1. Introduction

Belief graphical models, such as Bayesian networks [1], credal networks [2], or possibilistic networks [3] are powerful means to compactly represent uncertain information taking advantage of independence relationships. Despite the fact that they share many properties with probabilistic models, possibilistic networks have many interesting properties when modeling and reasoning with qualitative and partial information. For example, in a qualitative possibilistic framework, certain non-negligible gains can be made thanks to the idempotency property of the possibilistic operators (minimum and maximum) which can benefit the inference algorithms, as highlighted in [4]. Recent works [5–8] apply possibilistic techniques to semantic Web. Thus, possibility theory [9,10] is a natural alternative to represent and reason with some types of uncertainties. It is particularly suitable when only the ordering of plausibility between events makes sense. There are two main interpretations of possibility theory. The first one, based on the minimum operator, is called qualitative possibility theory. Here, the unit interval $[0, 1]$, used to weight the degrees of plausibility of events, is an ordinal scale. The second interpretation, called quantitative possibility theory, is based on the product operator. Here, the possibilistic interval $[0, 1]$ is used in the general sense (as in standard probabilities).

Inference in possibilistic networks has been extensively studied and many algorithms have emerged. On the other hand, while the results of the computational complexity of inference in Bayesian and probabilistic networks in general are well established [11–13], there is no such deep study for possibilistic networks. This paper aims at filling this gap. More precisely, this paper provides results representing additional advantages of possibilistic models in terms of computational complexity of inference and query answering [9, 10].

Essentially, in graphical models there are three common types of queries: computing most probable (or plausible) explanation (*MPE*); computing a posteriori probability (or possibility) degrees (*Pr*); and computing the maximum a posteriori explanation (*MAP*). These tasks are known to be very hard in the probabilistic setting. Indeed, the decision problems associated to *MPE*, *Pr*, *MAP* are *NP*-complete, *PP*-complete and NP^{PP}-complete respectively (see [12,13] for further details on the computational complexity of inference in probabilistic graphical models).

This paper[a] focuses on most plausible explanation (*MPE*) and maximum a posteriori (*MAP*) queries in the context of qualitative and quantitative possibilistic networks. One of the main results is to show that querying possibilistic networks is less costly than querying their probabilistic counterparts. More specifically, we show that the decision problems corresponding to the *MAP* and *MPE* queries in possibilistic models is *NP*-complete. The proofs, built incrementally, are provided for qualitative and quantitative models. The hardness of the decision problem of *MAP* (*resp. MPE*) in possibilistic networks, is shown on a special type of possibilistic networks called binary and Boolean possibilistic networks. Thus, we provide a reduction from 3SAT to a *MAP* query (*resp. MPE*) on a binary and Boolean possibilistic network. Finally, we provide reductions from querying possibilistic graphical models to two well-known *NP*-complete problems: weighted SAT and MaxSAT decision problems.

The remainder of this paper is structured as follows : Section 2 provides basic

[a]This paper is an extended version of a conference paper presented at ICTAI'18 [14]

concepts on possibilistic networks. Section 3 focuses on motivations and reviews related works. Section 4 introduces the definition of *MAP* and *MPE* inference in possibilistic networks and gives our first results on the computational complexity of these inferences. Section 5 presents an overview of our solution to prove the complexity results of the decision problems considered in this paper. The remaining sections provide the polynomial reductions used in this paper.

2. A brief refresher on possibility theory and possibilistic networks

In this section, we recall the main concepts of possibility theory and possibilistic networks (for further details, refer to [9] [15–17]). One of the main building blocks of possibility theory is the concept of a possibility distribution denoted by π. It is a mapping from the universe of discourse Ω (a finite and discrete set of all states of the world) to the unit interval $[0,1]$. By convention, a state $\omega \in \Omega$ such that $\pi(\omega)=1$ denotes that ω is fully possible while $\pi(\omega)=0$ means that it is impossible that ω is the real world. If there is at least one state $\omega \in \Omega$ such that $\pi(\omega)=1$, then the possibility distribution π is said to be normalized.

Given a possibility distribution π, one can define a possibility measure, defined for each event (subset of states) $\phi \subseteq \Omega$, by:

$$\Pi(\phi) = \max\{\pi(\omega) : \omega \in \phi\}. \tag{1}$$

$\Pi(\phi)$ assesses to what extent ϕ is coherent (compatible) with available information encoded by π.

Possibility degrees can be interpreted either quantitatively (product-based interpretation) like in probability theory or qualitatively (min-based interpretation) which considers degrees on an ordinal scale. Hence, the two interpretations lead to different ways to deal with possibility degrees. In particular, conditioning, namely updating the current beliefs given a new evidence, differs depending on the quantitative or qualitative interpretation of possibility degrees. We denote the min-based conditioning by $|_m$ [9, 18] and it is defined as follows : Given a possibility distribution π encoding the current beliefs, and a new evidence $\phi \subseteq \Omega$ (with $\Pi(\phi) > 0$), the conditional distribution $\pi(.|_m \phi)$ is obtained following Equation 2:

$$\pi(\omega_i|_m \phi) = \begin{cases} 1 & \text{if } \pi(\omega_i) = \Pi(\phi) \text{ and } \omega_i \in \phi; \\ \pi(\omega_i) & \text{if } \pi(\omega_i) < \Pi(\phi) \text{ and } \omega_i \in \phi; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The product-based conditioning is denoted by $|_*$ and it is defined as follows :

$$\pi(\omega_i|_* \phi) = \begin{cases} \dfrac{\pi(\omega_i)}{\Pi(\phi)} & \text{if } \omega_i \in \phi; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We will simply write $\pi(\omega|\phi)$ to indifferently refer to $\pi(\omega|_m\phi)$ or $\pi(\omega|_*\phi)$ when there is no ambiguity.

A possibility distribution can be compactly represented in the form of a graphical model also known as a possibilistic network. As in probabilistic graphical models, a possibilistic network denoted $\mathcal{PN} =< G, \Theta >$ is composed of two components:

- A directed acyclic graph (DAG) G where each node represents a discrete variable (from the set of variables $V = \{X_1, .., X_n\}$) and edges encode independence relations between variables.
- A set of local normalized possibility distributions Θ including a local table $\Theta_i = \pi_{\mathcal{PN}}(X_i|par(X_i))$ of each node X_i given its parents $par(X_i)$. The normalization condition on local possibility distributions is defined by:

$$\forall u_{ij} \in D_{par(X_i)} \max_{x_i \in D_{X_i}} \pi_{\mathcal{PN}}(x_i|u_{ij}) = 1.$$

At the semantical level, a possibilistic network encodes a unique joint possibility distribution obtained using the so-called chain rule. Depending on the used conditioning, there are also two definitions of the chain rule that can be used to derive a joint distribution underlying a possibilistic network. Let us denote by $\mathcal{PN}_m$ (respectively $\mathcal{PN}_*$) a min-based (respectively a product-based) possibilistic network. Then, the chain rule for these possibilistic networks is defined as:

$$\pi_{\mathcal{PN}_m}(X_1, .., X_n) = \min_{i=1,..,n} \pi_{\mathcal{PN}_m}(X_i|_m\ par(X_i)).$$

$$\text{and} \tag{4}$$

$$\pi_{\mathcal{PN}_*}(X_1, .., X_n) = \prod_{i=1,..,n} \pi_{\mathcal{PN}_*}(X_i|_*\ par(X_i)).$$

where $\prod$ is the product operator.

Example 2.1. In Figure 1, we find an example of a possibilistic network over four Boolean variables $V = \{A, B, C, D\}$. On this figure, the domain of each variable X involves only two values denoted x and $\neg x$.

For the sake of simplicity and when there is no ambiguity, $\mathcal{PN}$ will refer indifferently to $\mathcal{PN}_m$ or $\mathcal{PN}_*$.

3. Motivations and related works

Possibilistic networks may offer some advantages over probabilistic graphical models especially when it comes to modeling and reasoning with qualitative and partial uncertainty. Moreover, possibilistic netwoks may also offer nice features regarding practical and computational issues. This section provides two illustrations of nice features when it comes to modeling complex problems.

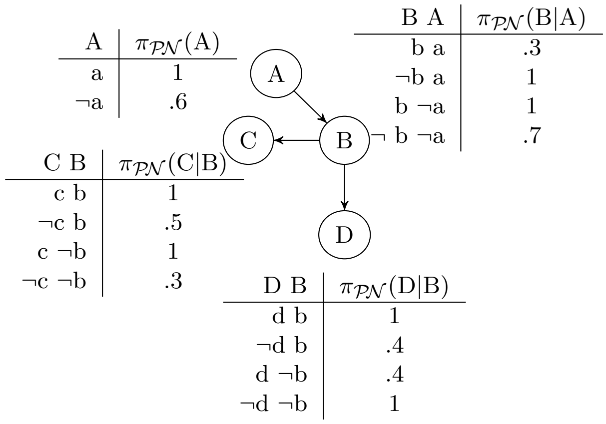


Fig. 1. Example of a possibilistic network $\mathcal{PN}$ over four boolean variables.

3.1. *The problem of probability underflow or undistinguishable likelihoods*

Many real-world problems (such as forecasting [19], simulation of physical [20] or biological systems [21, 22], etc.) need to model a sequential or a dynamic system involving many variables over long time periods. In this case, inference typically comes down to computing the likelihood of an outcome or any event of interest given some inputs. The problem then is that inferences for long sequences lead inevitably to what is called *probability underflow* problem due to propagating a long series of small probabilities (indeed, the computer representation of real numbers does not allow to represent extremely small probabilities and rounds them to zero). Consequently, some events with relatively different likelihoods will be associated to the same likelihoods. A very commonly used alternative is to rely on log likelihoods instead of computing the likelihood itself but then over long sequences this can lead to the overflow problem. Thanks to the use of idempotent operators such as the maximum and minimum, possibilistic propagation will not encounter such issues.

3.2. *High computational complexity*

It is well-known that, in the general case, inference in probabilistic models is a hard task. More precisely, the decision problem associated with *MAP* is NP^{PP}-complete. The interested reader can refer to [12, 13] for more details on complexity issues in Bayesian and credal networks. As mentioned in the introduction, it is worth to note that while complexity studies of inference in probabilistic graphical models are well-established [11], there is to the best of our knowledge no systematic study of inference issues in possibilistic networks (except a study of complexity of some inference tasks in possibilistic influence diagrams [16]). Of course, some propagation algorithms for probabilistic networks have already been adapted to the possibilistic setting and seem to show the same complexity. Among the seminal works on inference in possibilistic graphical models we find [23] focusing on inference in hypergraphs. Apart form that, most of the works are more or less direct adaptations of probabilistic graphical model inference algorithms. For instance, an elimination variable algorithm can be found in [24] in the context of possibilistic network classifiers while in [15], we find a possibilistic counterpart of the Message passing algorithm. In [25], the authors propose a direct adaptation of the Junction tree algorithm for the possibilistic setting. Possibilistic models could be used to approximate inference in some imprecise probabilistic models as proposed in [26] where an approach based on probability-possibility transformations is used to perform approximate *MAP* inference in credal networks. Clearly, modeling and reasoning with complex problems involving many variables will not be tractable unless strong assumption are made regarding the structure of the graphs. In this paper, one of the main results is to show that querying possibilistic graphical models has a lower computational complexity than querying probabilistic ones.

4. Possibilistic networks : Reasoning and inference

In this paper, we investigate two of the most common types of queries when reasoning with graphical models, that are *MAP* inference and *MPE* inference. *MAP* queries search for the most plausible instantiation of query variables Q given an evidence e (an instantiation of a set of variables E) while *MPE* queries search for the most plausible explanation of an evidence e. More formally,

MAP **query**: Let $\mathcal{PN}$ be a possibilistic graphical model over a set of variables V. Let also $Q{\subset}V$ be a subset of query variables and $E{\subset}V$ be a subset of evidence variables such that Q and E are disjoint subsets (namely, $Q{\cap}E{=}\emptyset$). Given an evidence $E{=}e$, the objective is to compute the most plausible instantiation q of Q, namely *MAP* queries aim to search for

$$\underset{q \in D_Q}{\operatorname{argmax}}(\Pi_{\mathcal{PN}}(q|_{\otimes}e)) \tag{5}$$

where $|_{\otimes}$ is either min-based or product-based conditioning.

MPE **query**: Let $\mathcal{PN}$ be a possibilistic network over the set of variables V, $E \subset V$ be a set of evidence variables. We denote X the set of remaining variables ($X = V \setminus E$). Then, given an evidence $E{=}e$, *MPE* query compute the most plausible instantiation x of X compatible with the evidence e. Namely[b]:

$$\underset{x \in X}{\operatorname{argmax}}(\Pi_{\mathcal{PN}}(x, e)). \tag{6}$$

[b]Note that $\Pi_{\mathcal{PN}}(x, e)$ is the possibility degree of the conjunction of x and e, especially since $X \cap E = \emptyset$. Another notation commonly used is $\Pi_{\mathcal{PN}}(x \wedge e)$.

In the case of a *MAP* query, the problem can be reduced to finding the most plausible assignment of query variables Q compatible with the evidence e. More precisely, using the maxitivity property of possibility theory allows to rewrite Equation (5) as follows:

$$\underset{q \in D_Q}{\operatorname{argmax}}(\Pi_{\mathcal{PN}}(q,e)). \tag{7}$$

This is formally stated in the following proposition.

Proposition 4.1. *Given a possibilistic network $\mathcal{PN}$, a set of query variables Q and an evidence e (instance of variables E), we have:*

$$\underset{q \in D_Q}{\operatorname{argmax}}(\Pi_{\mathcal{PN}}(q|e)) = \underset{q \in D_Q}{\operatorname{argmax}}(\Pi_{\mathcal{PN}}(q,e)). \tag{8}$$

for both min*-based and product-based conditioning rules.*

Proof.

- Let us start with the min-based conditioning. Given a possibilistic network $\mathcal{PN}_m$ over V and let Q and E be two subsets of V (*s.t.* $Q \cap E = \emptyset$). Then, computing $\operatorname{argmax}_{q \in D_Q}(\Pi(q|e))$ is equivalent to searching the instantiation q such that $\Pi(q|e) = 1$. By definition of the min-based conditioning, $\Pi(q|e) = 1$ if $\Pi(q,e) = \Pi(e)$. Assume that $\operatorname{argmax}_{q \in D_Q}(\Pi(q,e))$ is q' then since $\Pi(e) = \max_{\omega \models e} \pi(\omega)$ or said otherwise $\Pi(e) = \max_{q \in D_Q} \Pi(q,e)$ which is given by $\Pi(q',e)$.
- Let us now consider product-based conditioning. In the same way, since the possibilistic network $\mathcal{PN}_*$ is normalised then $\forall e \in E$, $\operatorname{argmax}_{q \in D_Q}(\Pi(q|e))$ is equivalent to searching the instantiation q such that $\Pi(q|e) = 1$. Which, by definition, is given by $\Pi(q|e) = \dfrac{\Pi(q,e)}{\Pi(e)}$, therefore, $\Pi(q|e) = 1$ if $\Pi(q,e) = \Pi(e)$. From there, assume that $\operatorname{argmax}_{q \in D_Q}(\Pi(q,e))$ is q' then since $\Pi(e) = \max_{\omega \models e} \pi(\omega) = \Pi(q',e)$. Thus, $\operatorname{argmax}_{q \in D_Q}(\Pi_{\mathcal{PN}}(q|e)) = \operatorname{argmax}_{q \in D_Q}(\Pi_{\mathcal{PN}}(q,e))$. □

Given this equivalence, we can focus only on the *MAP* problem redefined by Equation (7).

5. General overview of the contribution

In order to analyse the computational complexity of inference in possibilistic networks, we provide first in this section, a reminder of the notions of boolean satisfiability decision problems and a description of the different steps we will take, to prove that *MAP* inference (*resp. MPE* inference) is *NP*-complete in possibilistic networks. In particular, the analysis breaks down into showing the hardness and the completeness of the decision problems associated to *MAP* and *MPE* queries. Let us first denote each of these decision problems. More precisely[c],

- We denote by $\pi_{\otimes}$**-D-MAP**$(\mathcal{PN}_{\otimes}, Q, e, t)$ the decision problem associated to a *MAP* query in a possibilistic network (*i.e.* π_***-D-MAP**$(\mathcal{PN}_*, Q, e, t)$ in product-based possibilistic networks and π_m**-D-MAP**$(\mathcal{PN}_m, Q, e, t)$ in min-based possibilistic networks)
- We denote by $\pi_{\otimes}$**-D-MPE**$(\mathcal{PN}_{\otimes}, e, t)$ the decision problem associated to a *MPE* query in a possibilistic network (*i.e.* π_***-D-MPE**$(\mathcal{PN}_*, e, t)$ in product-based possibilistic networks and π_m**-D-MPE**$(\mathcal{PN}_m, e, t)$ in min-based possibilistic networks)

We will also refer to a special case of possibilistic networks involving only boolean variables (for variable domains) and binary possibility degrees 0 or 1 (namely, each conditional event is either fully possible or fully impossible). This type of belief networks is called in this paper Boolean and Binary possibilistic networks and they are denoted by B&B possibilistic networks. A joint B&B possibility distribution is therefore a particular case of a general possibility distribution which is defined over {0,1} rather than over then whole unit interval $[0, 1]$. Thus it keeps the same properties and the same definition of computations of conditioning and chain rules. The following introduces notations associated with *MAP* and *MPE* decision problems defined for B&B possibilistic networks:

- We denote by **B&B**$_{\otimes}$**-D-MAP**$(\mathcal{PN}_{B\&B_{\otimes}}, Q, e)$ the decision problem associated to *MAP* querying a binary and boolean possibilistic network.
- In the same way, we denote by **B&B**$_{\otimes}$**-D-MPE**$(\mathcal{PN}_{B\&B_{\otimes}}, e)$ the decision problem associated to *MPE* querying a binary and boolean possibilistic network.

We recall that the operator $\otimes$ can be either the min or product operation.

To show hardness and completeness of *MAP* and *MPE* queries, we will provide polynomial-time reductions from some known *NP*-complete problems to our *MAP* decision problems (*resp. MPE* decision problems) and conversely.

5.1. *A brief refresher on satisfiability problems*

Let us start with recalling the basic notions of boolean satisfiability. We only focus on formulas in conjunctive normal form since this is enough for the purpose of our study. Let us assume a set of boolean variables $V = \{X_1, ..., X_n\}$ and let us denote by x_i (resp. $\neg x_i$) the positive literal (resp. the negative literal) of variable X_i. A clause C is defined as a disjunction of literals. For example, a clause C could be: $x_1 \vee \neg x_2$.

[c]these decision problems will be formally defined in relevant sections

Definition 5.1. A conjunctive normal form formula (CNF) Ψ as a conjunction of clauses.

The formula $(x_1 \vee \neg x_2) \wedge (x_3 \vee \neg x_2)$ is an example of a CNF. A 3CNF is a conjunctive normal form formula where each clause is a disjunction of at most 3 literals.

A CNF formula Ψ is said satisfiable or consistent if there exists an assignment (also called interpretation) of all the variables that makes Ψ true. For CNF formulas, the boolean satisfiability decision problem CNF-SAT, denoted by **D-SAT**, is stated as follows:

Definition 5.2. We denote by **D-SAT**(Ψ) the decision problem associated to deciding whether there exists an assignment of variable that satisfies Ψ. It is stated by:
Input: The input is a formula Ψ given in a conjunctive normal form.
Question: The question is whether the formula Ψ is satisfiable ?

In the sequel, the **D-3SAT** decision problem is stated as follows :

Definition 5.3. We denote by **D-3SAT**(Ψ) the decision problem stated by:
Input: The input is a 3CNF formula, denoted by Ψ
Question: The question is whether the formula Ψ satisfiable ?

Example 5.1. Let us assume the set of variables $V = \{X_1, X_2, X_3, X_4\}$ and the following 3CNF Ψ on V:

$$(x_1 \vee \neg x_2 \vee x_3) \wedge$$
$$(\neg x_3 \vee \neg x_2 \vee x_4)$$

It is easy to check that Ψ is satisfiable. In fact, the assignment $\omega = x_1, x_2, \neg x_3, \neg x_4$ satisfies all clauses and makes Ψ take the truth value $True$. Hence, the answer to the decision problem **D-SAT**(Ψ) is "yes".

The other problem referred to in this paper is the weighted MaxSAT problem which generalizes the SAT problem as follows : Given a formula with non-negative integer weights for each clause, the task is to find an assignment of variables that maximizes the sum of the weights of the satisfied clauses. We the define associated decision problem for weighted MaxSAT as follows :

Definition 5.4. We denote by **D-WMaxSAT**(Ψ, k) the decision problem defined by:
Inputs:

- Ψ: a weighted CNF formula over $V = \{X_1, ..., X_n\}$ represented by

$$\Psi = \left\{ \begin{array}{c} (C_1, \alpha_1), \\ (C_2, \alpha_2), \\ ... \\ (C_m, \alpha_m). \end{array} \right\}$$

where $C_i's$ are clauses and $\alpha_i's$ are positive integers.

- k: a positive integer

Question: Is there an instantiation of variables V such that the sum of weights of satisfied clauses in Ψ is greater or equal to k?

Example 5.2. Let us assume the following weighted CNF formula Ψ over $V = \{X_1, X_2, X_3, X_4\}$:

$$\Psi = \left\{ \begin{array}{r} (x_1 \vee \neg x_2, 4), \\ (\neg x_1 \vee x_2, 6), \\ (\neg x_3 \vee \neg x_2 \vee x_4, 5), \\ (x_5 \vee x_4 \vee \neg x_1, 2) \end{array} \right\}$$

Let us assume k=10. The instantiation of the variables V $\omega = x_1, \neg x_2, x_3, x_4$ satisfies all clauses except $(\neg x_1 \vee x_2, 6)$. Hence $\sum\{\alpha_i : (C_i, \alpha_i) \in \Psi \text{ s.t } \omega \models C_i\} = 11 \geq 10$ where $\models$ denotes the propositional logic satisfaction relation. Consequently, the answer to the decision problem **D-WMaxSAT**$(\Psi, 10)$ is "yes".

5.2. *Steps of the solution*

In the following, we provide the proof of the *NP*-completeness of $\pi_\otimes$-**D-MAP** and $\pi_\otimes$-**D-MPE** decision problems. This is done through the following steps:

- We first show the *NP*-hardness of π_m-**D-MAP** and π_*-**D-MAP**. We will provide a reduction from the **D-3SAT** decision problem to both π_m-**D-MAP** and π_*-**D-MAP** decision problems. In this reduction, we use the restricted version, B&B possibilistic networks, and we will provide intermediary results and the reductions from the **D-3SAT** decision problem to **B&B**$_\otimes$-**D-MAP** decision problem.
- We provide a reduction of the π_m-**D-MAP** decision problem, defined for min-based possibilistic networks, to the **D-SAT** decision problem (for completeness in min-based possibilistic networks).
- We provide the completeness of the proof by reducing the π_*-**D-MAP** decision problem, defined for product-based possibilistic networks, to the **D-WMaxSAT** decision problem.

This concludes the proof for *MAP* querying possibilistic networks. To tackle the *MPE* querying of possibilistic networks, we will follow the same steps:

- We show the *NP*-hardness of π_m**-D-MPE** and π_***-D-MPE** with a reduction from the **D-3SAT** decision problem to **B&B**$_\otimes$**-D-MPE** decision problem.
- We provide a reduction of the π_m**-D-MPE** decision problem, defined for min-based possibilistic networks, to the **D-SAT** decision problem (for completeness in min-based possibilistic networks).
- Lastly, we will focus on reducing the π_***-D-MPE** decision problem, defined for product-based possibilistic networks, to the **D-WMaxSAT** decision problem (for completeness in product-based possibilistic networks).

6. Analysis of *MAP* querying a possibilistic network

In this section, we focus on proving that the decision problem behind *MAP* inference in possibilistic networks is *NP*-complete. First, we propose, in Subsection 6.1, to reduce the 3SAT decision problem to *MAP* querying B&B possibilistic networks. This shows that the decision problem behind *MAP* is *NP*-hard. By proving, in Subsections 6.2 and 6.3, that the decision problem associated to *MAP* inference is also in *NP*, hence we prove that *MAP* inference is *NP*-complete.

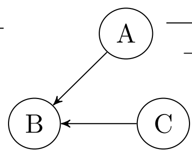
6.1. *From the 3SAT problem to the one of MAP querying in B&B possibilistic networks*

In this case, we face only two kinds of queries: Given an evidence e (instantiation of variables E), the question is *is there an instantiation q of query variables Q such that $\Pi_{\mathcal{PN}_\otimes}(q \wedge e) \geq 0$ or such that $\Pi_{\mathcal{PN}_\otimes}(q \wedge e) \geq 1$ with $\otimes$=m (resp. $\otimes$=$*$) for* min*-based (resp. product-based) possibilistic setting ?* The inequality $\Pi_{\mathcal{PN}_\otimes}(q \wedge e) \geq 0$ is satisfied trivially since any instantiation q of Q is an answer to the query.

Therefor, we will only focus on studying the computational complexity of decision problems π_m**-D-MAP**$(\mathcal{PN}_{B\&B_m}, Q, e, 1)$ and π_***-D-MAP**$(\mathcal{PN}_{B\&B_*}, Q, e, 1)$.

Example 6.1. We illustrate the decision problem π**-D-MAP**$(\mathcal{PN}_{B\&B}, Q, e, 1)$ on the B&B possibilistic network of Figure 2 over the boolean variables $V = \{A, B, C\}$.

B A C	$\pi_{B\&B}$(B|AC)
b a c	1
¬b a c	0
b a ¬c	1
¬b a ¬c	1
b ¬a c	1
¬b ¬a c	0
b ¬a ¬c	0
¬b ¬a ¬c	1



A	$\pi_{B\&B}$(A)
a	1
¬a	1

C	$\pi_{B\&B}$(C)
c	1
¬c	0

Fig. 2. Example of a Boolean and Binary possibilistic network.

Let $Q = \{B\}$ be the set of query variables and $E = \{C\}$ be the set of evidence variables. Assume that $e = c$, then one can check that the answer to the question: is there an instantiation q of B such that $\Pi_{\mathcal{PN}_{B\&B}}(q \wedge c) = 1$? is "yes". Indeed, we have $\Pi_{\mathcal{PN}_{B\&B}}(bc) = 1$ and this is valid independently if we consider the min-based chain rule or the product-based chain rule.

6.1.1. *Equivalence between the MAP decision problem in* min*-based B&B networks and product-based B&B networks*

Given the definition of a B&B possibilistic network, Proposition 6.1 states that the decision problems π_*-**D-MAP**$(\mathcal{PN}_{B\&B_m}, Q, e, 1)$ and π_*-**D-MAP**$(\mathcal{PN}_{B\&B_*}, Q, e, 1)$ are equivalent.

Proposition 6.1. *Let e be an evidence and Q be a subset of query variables. Let also $\mathcal{PN}_{B\&B_m}$ and $\mathcal{PN}_{B\&B_*}$ be two B&B possibilistic networks where $\forall X_i$, $\forall \mu$ an instance of parents of X_i, $\pi_{\mathcal{PN}_{B\&B_m}}(X_i|\mu) = \pi_{\mathcal{PN}_{B\&B_*}}(X_i|\mu)$. Then the answer to the question π_m-**D-MAP**$(\mathcal{PN}_{B\&B_m}, Q, e, 1)$ is "yes" iff the answer to the question π_*-**D-MAP**$(\mathcal{PN}_{B\&B_*}, Q, e, 1)$ is "yes".*

Following Proposition 6.1, the answer to a *MAP* query in a B&B possibilistic network does not depend on whether we are considering the min-based version of B&B possibilistic networks or the product-based one. The proof of Proposition 6.1 is straightforward and it is based on the fact that operators $*$ and min when only applied to possibility degrees 0 and 1 lead to the same results. Hence, when one uses only binary degrees $\{0, 1\}$, then the joint possibility distributions associated with $\mathcal{PN}_m$ and $\mathcal{PN}_*$ are equal. Namely:

Proposition 6.2. *Let $\mathcal{PN}_{B\&B_m}$ and $\mathcal{PN}_{B\&B_*}$ be two B&B possibilistic networks such that $\forall X_i$, $\forall \mu$ an instance of parents of X_i, $\pi_{\mathcal{PN}_{B\&B_m}}(X_i|\mu) = \pi_{\mathcal{PN}_{B\&B_*}}(X_i|\mu)$. Then we have:*

$$\forall \omega \in \Omega, \pi_{\mathcal{PN}_{B\&B_m}}(\omega) = \pi_{\mathcal{PN}_{B\&B_*}}(\omega). \tag{9}$$

The proof of Proposition 6.2 is also straightforward and it directly follows from the fact that if a and b are either equal to 0 or 1 then $\min(a, b) = a * b$. Consequently, the use of min-based chain rule or product-based chain rule leads to same results.

6.1.2. *Definition of the B&B network associated to a 3CNF*

We can now start addressing the reduction from the 3SAT problem to querying B&B possibilistic networks. As we have already showed that *MAP* querying B&B networks is the same in min-based or in product-based B&B ones, we can assume in this section the decision problem in the general case, denoted by **B&B-D-MAP**. Since $\Pi_{\mathcal{PN}}(q \wedge e) \geq 1$ is equivalent to $\Pi_{\mathcal{PN}}(q \wedge e) = 1$ then there is no need to specify the threshold t. We get the following definition :

Definition 6.1. We denote by **B&B-D-MAP**($\mathcal{PN}_{B\&B}, Q, e$) the decision problem associated with *MAP* querying a B&B possibilistic network that we define as follows:
Inputs: The input of this decision problem has three elements :

- $\mathcal{PN}_{B\&B}$: a B&B possibilistic network over V=$\{X_1, ..., X_n\}$
- e (evidence): an instantiation of a set of evidence variables E
- Q (query): a set of query variables such that $Q \cap E = \emptyset$

Question: Is there an instantiation q of variables Q such that $\Pi_{\mathcal{PN}_{B\&B}}(q \wedge e)$=1?

The first thing we provide is the B&B possibilistic network associated to a given 3CNF formula Ψ. Our reduction is inspired from the reduction provided in [27] to show the NP-hardness of probabilistic inference in belief networks. The B&B network associated to a given 3CNF formula is given in Definition .

Definition 6.2. Let $\Psi = C_1 \wedge C_2 \wedge ... \wedge C_m$ be a 3CNF formula. Let $V = \{X_1, ..., X_n\}$ be the set of propositional variables appearing in Ψ. The B&B possibilistic network associated with Ψ, denoted by $\mathcal{PN}_\Psi$ is defined as follows:

(1) **Representing propositional variables**: For each propositional symbol X_i involved in Ψ, we create a rooted boolean node variable, also and simply denoted by X_i, in the graph (with two values x_i and $\neg x_i$). Each rooted variable X_i is associated with the local uniform possibility distribution : $\pi_{\mathcal{PN}_\Psi}(x_i)$=1 and $\pi_{\mathcal{PN}_\Psi}(\neg x_i)$=1.
(2) **Encoding the satisfaction of a clause** C_j: For each clause C_j of Ψ, we introduce a conditional node variable, denoted C_j. C_j is a boolean variable whose two values are denoted c_j and $\neg c_j$. The parents of C_j are the rooted variables X_i that are involved in C_j. Each conditional node C_j is associated with a conditional possibility distribution given by: $\forall u_{jk}$ an instance of parents of C_j:

$$\pi_{\mathcal{PN}_\Psi}(c_j|u_{jk}) = \begin{cases} 1, \text{ if } u_{jk} \models C_j, \\ 0, \text{ otherwise.} \end{cases}$$

$$\pi_{\mathcal{PN}_\Psi}(\neg c_j|u_{jk}) = \begin{cases} 0, \text{ if } u_{jk} \models C_j, \\ 1, \text{ otherwise.} \end{cases}$$

where u_{jk} is an instantiation of node C_j's parents, namely the instantiation of variables X_i involved in C_j and $u_k \models C_j$ means that the instantiation u_k satisfies the clause C_j.
(3) **Encoding the satisfaction of the 3CNF formula** Ψ: At the end, we introduce a single boolean node denoted E_Ψ to represent the satisfiability of the whole formula Ψ. Its values are denoted e_Ψ and $\neg e_\Psi$ and it has all nodes $C_j's$ as parents. The conditional possibility distributions associated with E_Ψ are given as follows :

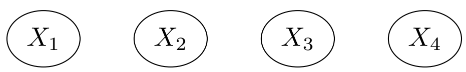
$$\pi_{\mathcal{PN}_\Psi}(e_\Psi|C_1 \wedge .. \wedge C_m) = \begin{cases} 1, \text{ if } \forall C_j,\ C_j = c_j, \\ 0, \text{ otherwise } (\exists j \in \{1..m\} \text{ s.t. } C_j = \neg c_j) \end{cases}$$

$$\pi_{\mathcal{PN}_\Psi}(\neg e_\Psi | C_1 \wedge .. \wedge C_m) = \begin{cases} 0, \text{ if } \forall C_j,\ C_j = c_j, \\ 1, \text{ otherwise} \end{cases}$$

Note that reducing from 3SAT clauses to a B&B possibilistic network given by Definition 6.2 is done in polynomial time. The space complexity is also polynomial with respect to the size of the input formula Ψ.

Example 6.2. Let us consider the 3CNF Ψ of Example 5.1.

According to Definition 6.2, the B&B possibilistic network $\mathcal{PN}_\Psi$ associated to Ψ involves three levels of nodes where the first level represents the set of variables. In this example, we have the first level involving the nodes X_1, X_2, X_3 and X_4 as shown on Figure 3.



X_1	
x_1	1
$\neg x_1$	1

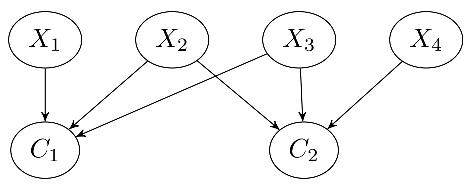
X_2	
x_2	1
$\neg x_2$	1

X_3	
x_3	1
$\neg x_3$	1

X_4	
x_4	1
$\neg x_4$	1

Fig. 3. First level of nodes in $\mathcal{PN}_\Psi$.

The second level of nodes contains 2 nodes C_1 and C_2 with local distributions as depicted on Figure 4. Note that in Figures 4 and 5, the entries of local distributions denoted by _ _ _ represent the remaining instantiations of $par(C_j)$ and $par(E_\Psi)$.

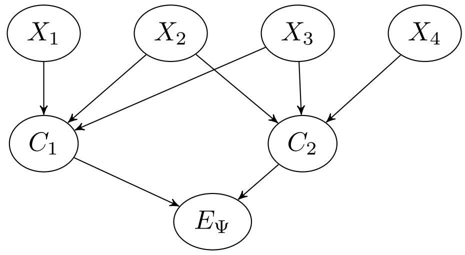


C_1	$X_1X_2X_3$	
c_1	$\neg x_1 x_2 \neg x_3$	0
c_1	_ _ _	1
$\neg c_1$	$\neg x_1 x_2 \neg x_3$	1
$\neg c_1$	_ _ _	0

C_2	$X_2X_3X_4$	
c_2	_ _ _	1
c_2	$x_2 x_3 \neg x_4$	0
$\neg c_2$	_ _ _	0
$\neg c_2$	$x_2 x_3 \neg x_4$	1

Fig. 4. First two levels of nodes X_i and C_j in $\mathcal{PN}_\Psi$.

When introducing the final node E_Ψ representing the whole 3CNF formula, we obtain the final binary possibilistic network of Figure 5.



E_Ψ	$C_1 C_2$	
e_Ψ	$c_1 c_2$	1
e_Ψ	- - -	0
$\neg e_\Psi$	$c_1 c_2$	0
$\neg e_\Psi$	- - -	1

Fig. 5. B&B network $\mathcal{PN}_\Psi$ obtained from the 3CNF formula Ψ given in Example 5.1.

6.1.3. *Reduction from the 3SAT problem to the **B&B-D-MAP** problem*

Theorem 6.1 gives the reduction from the decision problem **D-3SAT**(Ψ) into **B&B**$_m$**-D-MAP**($\mathcal{PN}_\Psi, Q, e$). The input e is let to e_Ψ while Q is set to the remaining variables in $\mathcal{PN}_\Psi$ (namely, $(\{X_1, ..., X_n\} \cup \{C_1, ..., C_m\}) \setminus \{E_\Psi\}$).

Theorem 6.1. Let Ψ be a 3CNF formula. Let also $\mathcal{PN}_\Psi$ be the B&B network given by Definition 6.2. Let $V_{\mathcal{PN}_\Psi}$ be the set of variables in $\mathcal{PN}_\Psi$, namely $\{X_1, ..., X_n\} \cup \{C_1, ..., C_m\} \cup \{E_\Psi\}$. Then, **D-3SAT**($\Psi$) answer is "yes" if and only if the **B&B**$_m$**-D-MAP**($\mathcal{PN}_\Psi, (V_{\mathcal{PN}_\Psi} \setminus \{E_\Psi\}), e_\Psi$) answers "yes" where **D-3SAT** is given in Definition 5.3 and **B&B**$_m$**-D-MAP** is given by Definition 6.1.

Proof.
− Let us first assume that the answer to **D-3SAT**(Ψ) is "yes", meaning that there exists an instantiation of the variables $\{X_1, ..., X_n\}$, denote by ω^*, satisfying all the clauses in Ψ. If ω is an interpretation and X is a variable then we simply denote by $\omega[X]$ the instance of X present in ω.

Let us build an interpretation, denoted $\omega_{\mathcal{PN}_\Psi}$, of $V_{\mathcal{PN}_\Psi}$ such that $\omega_{\mathcal{PN}_\Psi} \models e_\Psi$ and $\pi_{\mathcal{PN}_\Psi}(\omega_{\mathcal{PN}_\Psi}) = 1$. For the variable E_Ψ, we let $\omega_{\mathcal{PN}_\Psi}[E_\Psi] = e_\Psi$. For variables $X_i \in \{X_1, ..., X_n\}$ we let $\omega_{\mathcal{PN}_\Psi}[X_i] = \omega^*[X_i]$. For variables $C_j \in \{C_1, ..., C_m\}$ we simply let $\omega_{\mathcal{PN}_\Psi}[C_j] = c_j$. Now, let us show that indeed $\pi_{\mathcal{PN}_\Psi}(\omega_{\mathcal{PN}_\Psi}) = 1$.

Recall that for all the variables X_i in $\mathcal{PN}_\Psi$, we have $\pi_{\mathcal{PN}_\Psi}(X_i) = 1$. Since ω^* satisfies all the clauses, then for all variables C_j in $\mathcal{PN}_\Psi$ (namely, the set of nodes representing the clauses), we have $\pi_{\mathcal{PN}_\Psi}(c_j|u_{jk}) = 1$ where $\omega^* \models u_{jk}$. Lastly, the variable $E_\Psi = e_\Psi$ when all $C_j's$ are set to $c_j's$ respectively have a possibility degree of 1 ($\pi_{\mathcal{PN}_\Psi}(e_\Psi | c_1 \wedge ... \wedge c_m) = 1$).

Therefore, using the min-based chain rule, we have

$$\begin{aligned}\pi_{\mathcal{PN}_\Psi}(\omega_{\mathcal{PN}_\Psi}) &= \min\{\pi_{\mathcal{PN}_\Psi}(e_\Psi|c_1 \wedge ... \wedge c_m),\\ &\quad \min_{j=1,...,m,\omega_{\mathcal{PN}_\Psi}\models u_{c_j}} \pi_{\mathcal{PN}_\Psi}(c_j|u_{c_j}),\\ &\quad \min_{i=1,...,n,\omega_{\mathcal{PN}_\Psi}\models X_i} \pi_{\mathcal{PN}_\Psi}(X_i))\}\\ &= 1\end{aligned}$$

where u_{c_j} is the instance parents of C_j such that $\omega_{\mathcal{PN}_\Psi} \models u_{c_j}$. Therefore, defining q as the instantiation of Q satisfied by $\omega_{\mathcal{PN}_\Psi}$ we have $\Pi_{\mathcal{PN}_\Psi}(q \wedge e_\Psi) = 1$, hence **B&B**$_m$**-D-MAP**$(\mathcal{PN}_\Psi, (V_{\mathcal{PN}_\Psi} \setminus \{E_\Psi\}), e_\Psi)$ is "yes".

− Let us assume that the answer to **D-3SAT**(Ψ) is "no". Hence, whatever the considered interpretation $\omega_{\mathcal{PN}_\Psi}$ where $\omega_{\mathcal{PN}_\Psi} \models e_\Psi$ there exists at least C_j such that $\pi_{\mathcal{PN}_\Psi}(c_j|u_{c_j}) = 0$ with $\omega_{\mathcal{PN}_\Psi} \models u_{c_j}$. Hence, $\pi_{\mathcal{PN}_\Psi}(\omega_{\mathcal{PN}_\Psi}) = 0$. So using the min operator of the chain rule, we obtain that $\Pi_{\mathcal{PN}_\Psi}(q \wedge e_\Psi) = 0$ for all instantiation q of Q. Hence, **B&B**$_m$**-D-MAP**$(\mathcal{PN}_\Psi, (V_{\mathcal{PN}_\Psi} \setminus \{E_\Psi\}), e_\Psi)$ is "no". □

By this reduction we have shown that *MAP* querying possibilistic network is *NP*-hard. In addition to this proof, we provide the completeness of π_m**-D-MAP** and π_***-D-MAP**. One can either show their membership to *NP* or provide reductions from π_m**-D-MAP** and π_***-D-MAP** to SAT and WMAXSAT decision problems. In the following, we adopt the second option. Indeed, the proposed reductions can be used as useful transformations for implementation of MAP queries in possibilistic networks using SAT solvers.

6.2. *Reduction from MAP querying* min*-based possibilistic networks to the SAT problem*

In this subsection, we do not restrict ourselves to binary possibility distributions. Namely, possibility degrees can take values in the unit interval $[0, 1]$. However, for the sake of simplicity, we still only deal boolean variables. This is not a restriction since the proof can easily be extended by encoding a non-boolean variable by a set of boolean variables. We reduce here the decision problem π_m**-D-MAP** to the decision problem **D-SAT**.

It is time now to formally define the decision problem associated to a *MAP* query in min-based possibilistic networks, denoted π_m**-D-MAP**.

Definition 6.3. By π_m**-D-MAP**$(\mathcal{PN}_m, Q, e, t)$ we denote the decision problem associated with *MAP* querying *min*-based possibilistic networks that we define by:
Input: The input of this decision problem is composed of four elements :

- $\mathcal{PN}_m$: a min-based possibilistic network
- e (evidence): an instantiation of a set of variables E
- Q (query): a set of variables with $Q \cap E = \emptyset$
- t: a real number in $(0, 1]$.

Question: Is there an instantiation q of non observed variables Q such that $\Pi_{\mathcal{PN}_m}(q \wedge e) \geq t$?

6.2.1. *Definition of a CNF formula associated with a* min*-based possibilistic network*

We now define the reduction from a min-based possibilistic network $\mathcal{PN}_m$ into a CNF formula, denoted $\Psi_{\mathcal{PN}_m,Q,e,t}$. Definition 6.4 specifies the CNF formula associated to the network $\mathcal{PN}_m$, the set of query variables Q, the evidence e and the positive real number t in $\Psi_{\mathcal{PN}_m,Q,e,t}$.

Definition 6.4. Let $\mathcal{PN}_m$ be a min-based possibilistic network over the set of boolean variables V=$\{X_1, ..., X_n\}$. Let Q be a subset of V, $e = e_1, ..., e_l$ be an instantiation of evidence variables E (with $Q \cap E = \emptyset$) and let t be a threshold. Then $\Psi_{\mathcal{PN}_m,Q,e,t}$ over the same set of boolean variables $V = \{X_1, ..., X_n\}$, is given by:

$$\Psi_{\mathcal{PN}_m,Q,e,t} = \{(\neg x_i \vee \neg u_{ij}) : \pi_{\mathcal{PN}_m}(x_i|u_{ij}) < \mathbf{t}\} \\ \cup \{\mathbf{e_k} : \mathbf{k} = \mathbf{1}, ..., \mathbf{l}\}$$

It is obvious that this reduction is achieved in polynomial time and space with respect to the size of $\mathcal{PN}_m$.

Example 6.3. Let us use the possibilistic network $\mathcal{PN}_m$ of Figure 1 over the variables $V = \{A, B, C, D\}$. Let the set of evidence variables be E=$\{D\}$ and let $e = \{D = d\}$ be an instantiation of E, Q=$\{B, C\}$ be the query variables and $t = .5$. Using the transformation of Definition 6.4, we obtain the CNF $\Psi_{\mathcal{PN}_{m,\{B,C\},d,.5}}$ given in the following :

$$\Psi_{\mathcal{PN}_m,\{B,C\},d,.5} = \left\{ \begin{array}{c} (c \vee b) \wedge \\ (d \vee \neg b) \wedge \\ (\neg d \vee b) \wedge \\ (\neg b \vee \neg a) \wedge \\ d \end{array} \right\}$$

6.2.2. *Reduction from a* min*-based possibilistic network into a CNF*

Theorem 6.2 states that any π_m**-D-MAP** can be reduced to **D-SAT**.

Theorem 6.2. Let $\mathcal{PN}_m$ be a min-based possibilistic network, Q be the query variables, e be the evidence (an instantiation of variables E) and t be a real number in the interval $(0, 1]$. Let $\Psi_{PN_m,Q,e,t}$ be the CNF formula given by Definition 6.4. Then, the answer to π_m**-D-MAP**($\mathcal{PN}_m, Q, e, t$) is "yes" iff the answer to **D-SAT**($\Psi_{\mathcal{PN}_m,Q,e,t}$) is also "yes" where π_m**-D-MAP** is given by Definition 6.3 and **D-SAT** is given by Definition 5.2.

Proof.
− Let us assume that $\Psi_{\mathcal{PN}_m,Q,e,t}$ is satisfiable. This means that there exists an instantiation of all variables, denoted by ω^*, that satisfies all clauses of $\Psi_{\mathcal{PN}_m,Q,e,t}$ including $e = e_1, ..., e_l$. Recall that by construction of $\Psi_{\mathcal{PN}_m,Q,e,t}$, if $(\neg x_i \vee \neg u_{ij}) \in \Psi_{\mathcal{PN}_m,Q,e,t}$ then we have $\pi_{\mathcal{PN}_m}(x_i|u_{ij}) < t$. So if ω^* satisfies all clauses in $\Psi_{\mathcal{PN}_m,Q,e,t}$ then ω^* falsifies each of the formulas in $\{(x_i \wedge u_{ij}) : (\neg x_i \vee \neg u_{ij}) \in \Psi_{\mathcal{PN}_m,Q,e,t}\}$. This means that all conditionals $\pi_{\mathcal{PN}_m}(x_i|u_{ij})$ used in chain rule for defining $\pi_{\mathcal{PN}_m}(\omega^*)$ have a possibility degree greater or equal to t.Hence, their minimal is also greater or equal to t. Therefore, using the min-based chain rule we get $\pi_{\mathcal{PN}_m}(\omega^*) \geq t$.

Denoting now $q = \omega^*[Q]$ the instantiation of the variables Q such that $\omega^* \vDash q$, we have $\Pi_{\mathcal{PN}_m}(q \wedge e) \geq t$ since $\pi_{\mathcal{PN}_m}(\omega^*) \geq t$, $\omega^* \models q$ and $\omega^* \vDash e$. Hence the answer to π_m**-D-MAP**$(\mathcal{PN}_m, Q, e, t)$ is also "yes".

− Assume that $\Psi_{\mathcal{PN}_m,Q,e,t}$ is unsatisfiable. Then for all instantiation of variables ω such that $\omega \models e(= e_1 \wedge .. \wedge e_l)$, there exists at least a clause $C_i = \neg x_i \vee \neg u_{ij}$ that is falsified by ω (and hence $\omega \models x_i \wedge u_{ij}$). Now by construction of $\Psi_{\mathcal{PN}_m,Q,e,t}$, we have $\pi_{\mathcal{PN}_m}(x_i|u_{ij}) < t$, so using the min-based chain rule we have $\forall \omega \models e$, $\pi_{\mathcal{PN}_m}(\omega) < t$ and therefore $\forall q \in D_Q$, $\Pi_{\mathcal{PN}_m}(q \wedge e) < t$. □

Let us illustrate the above theorem with a *MAP* query.

Example 6.4. Let us assume the CNF formula $\Psi_{\mathcal{PN}_m,\{B,C\},d,.5}$, of Example 6.3, corresponding to the *MAP* query:

> Is there an instantiation q of query variables $\{B, C\}$ such that $\Pi_{\mathcal{PN}_m}(q \wedge e) \geq .5$?

Namely, the decision problem is π_m**-D-MAP**$(\mathcal{PN}_m, \{B, C\}, d, .5)$. There exist two models $\neg abcd$ and $\neg ab\neg cd$. Hence, the answer to **D-SAT**$(\Psi_{\mathcal{PN}_m,Q,e,t})$ is "yes". Finally, using the min-based chain rule on the network of Figure 1, one obtains $\pi(\neg abcd) = .6$; hence $\Pi_{\mathcal{PN}_m}(bcd) = .6$ which is higher or equal than .5. Then the answer to π_m**-D-MAP**$(\mathcal{PN}_m, \{B, C\}, d, .5)$ is "yes".

This proves that *MAP* querying a min-based possibilistic network is *NP*-complete. Let us now address the product-based possibilistic setting by providing a reduction from the decision problem π_***-D-MAP** to the decision problem **D-WMaxSAT**, given by Definition 5.4.

6.3. *From MAP querying product-based possibilistic networks to WMaxSAT*

In this section, we will consider that the possibility degrees in the possibilistic networks are of the form $2^{-\alpha_i}$ (plus 0 and 1) where α_i's are positive integers. Having uncertainty degrees of the form $2^{-\alpha_i}$ will allow us to easily reduce $\mathcal{PN}_*$ to WMaxSAT given the fact that the weights used in WMaxSAT are integers (it is enough to use $-\log_2(2^{-\alpha_i})$ to get positive integers). This assumption is done again for the sake of clarity but the proof can be generalized to other real numbers between 0 and 1. Note that α_i may represent a degree of surprise used in Spohn's ordinal conditional function [28].

Before giving the definition of the transformation, we formally define the decision problem associated to *MAP* querying a product-based possibilistic network π_***-D-MAP**.

Definition 6.5. By π_***-D-MAP**$(\mathcal{PN}_*, Q, e, t)$ we denote the decision problem associated with *MAP* querying product-based possibilistic networks that we define by:
Input: The input of this decision problem is composed of four elements :

- $\mathcal{PN}_*$: a product-based possibilistic network
- e (evidence): an instantiation of a set of variables E
- Q (query): a set of variables with $Q \cap E = \emptyset$
- t: a real number in $(0, 1]$.

Question: Is there an instantiation q of non observed variables Q such that $\Pi_{\mathcal{PN}_*}(q \wedge e) \geq t$?

6.3.1. *Definition of a weighted CNF formula associated to a product-based possibilistic network*

In the following definition, we give the weighted CNF formula associated with a *MAP* query in product-based possibilistic networks. More precisely, it takes into account the evidence $e = e_1, ..., e_l$ of the set of variables E (of size $|E| = l$), the set of query variables Q and the threshold t to produce the associated weighted CNF formula.

Definition 6.6. Let $\mathcal{PN}_*$ be a product-based possibilistic network over the set of boolean variables $V = \{X_1, ..., X_n\}$. Let Q be a subset of V, $e = e_1, ..., e_l$ be an instantiation of evidence variables E (with $Q \cap E = \emptyset$) and t be a threshold. Then $\Psi_{\mathcal{PN}_*,Q,e,t}$ is defined by: $\Psi_R \cup \Psi_0 \cup \Psi_e$ where

$$\begin{aligned} \Psi_R &= \{(\neg x_i \vee \neg u_{ij}, \alpha_i) : \pi_{\mathcal{PN}_*}(x_i|u_{ij}) = 2^{-\alpha_i}\}, \\ \Psi_0 &= \{(\neg x_i \vee \neg u_{ij}, M) : \pi_{\mathcal{PN}_*}(x_i|u_{ij}) = 0\}, \\ \Psi_e &= \{(e_k, M) : k = 1, ..., l\}, \end{aligned} \tag{10}$$

where M is a positive number such that $M > \sum\{\alpha_i : (\neg x_i \vee \neg u_{ij}, \alpha_i) \in \Psi_R\}$.

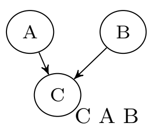
Ψ_R represents the clauses in $\Psi_{\mathcal{PN}_*,Q,e,t}$ such that have possibility degrees of the form $2^{-\alpha_i}$. Ψ_0 represents the clauses for which the possibility degrees in $\mathcal{PN}_*$ are 0. The information Ψ_e represents the clauses added to enforce the evidence. Intuitively, the integer weight M is used for fully certain pieces of information. Besides, $\Psi_0 \wedge \Psi_e$ is of course assumed to be consistent (this reflects the very reasonable assumption that the evidence is somewhat possible).

For the following, we will also denote by $X = \sum\{\alpha_i : (\neg x_i \vee \neg u_{ij}, \alpha_i) \in \Psi_R\}$ the sum of weights in Ψ_R.

Example 6.5 illustrates Definition 6.6.

Example 6.5. Let us consider the product-based possibilistic network $\mathcal{PN}_*$ of Figure 6. Let $Q = \{B\}$ be a subset of V, let $e = \neg c$ be an instantiation of evidence variables $E = \{C\}$ and let $t = 2^{-2}$ be the threshold.

A	$\pi_{\mathcal{PN}_*}$(A)
a	1
¬a	2^{-4}



B	$\pi_{\mathcal{PN}_*}$(B)
b	2^{-8}
¬b	1

C A B	$\pi_{\mathcal{PN}_*}$(C|AB)
c a b	2^{-7}
¬c a b	1
c a ¬b	1
¬c a ¬b	2^{-2}
c ¬a b	0
¬c ¬a b	1
c ¬a ¬b	0
¬c ¬a ¬b	1

Fig. 6. Example of a product-based possibilistic network $\mathcal{PN}_*$ over A, B and C.

Let $M = 30$. Then following Definition 6.6, the weighted CNF formula $\Psi_{\mathcal{PN}_*,\{B\},\neg c,2^{-2}}$ is

$$\Psi_{\mathcal{PN}_*,\{B\},\neg c,2^{-2}} = \left\{ \begin{array}{ll} \left.\begin{array}{c} (a,4), \\ (\neg b,8), \\ (\neg c \vee \neg a \vee \neg b, 7), \\ (c \vee \neg a \vee b, 2), \end{array}\right\} & \Psi_R \\ \left.\begin{array}{c} (\neg c \vee a \vee b, 30), \\ (\neg c \vee a \vee \neg b, 30), \end{array}\right\} & \Psi_0 \\ \left.\begin{array}{c} (\neg c, 30) \end{array}\right\} & \Psi_e \end{array} \right\}$$

6.3.2. *Reduction from a product-based possibilistic network to a weighted CNF formula*

Theorem 6.3 provides the reduction from the decision problem π_***-D-MAP**$(\mathcal{PN}_*, Q, e, t)$ into **D-WMaxSAT** $(\Psi_{\mathcal{PN}_*,Q,e,t}, k)$. We will denote by Z the number of possibility degrees, $\pi_{\mathcal{PN}_*}(x_i|u_{ij})$ in $\mathcal{PN}_*$ that are equal to 0 (namely, Z is the number of clauses in Ψ_0).

The input k is let to $X + \log_2 t + M * (Z + |E|)$ while $\Psi_{\mathcal{PN}_*,Q,e,t}$ is the weighted CNF formula given associated to $\mathcal{PN}_*$ given by Definition 6.6 (we also assume for only sake of simplicity that t is of the form $2^{-\alpha}$ with α an integer). More formally:

Theorem 6.3. Let $\mathcal{PN}_*$ be a product-based possibilistic network. Let Q be a subset of V, e be an instantiation of variables E and t be a threshold. Let $\Psi_{\mathcal{PN}_*,Q,e,t}$ be the CNF formula given by Definition 6.6. Then, π_***-D-MAP**$(\mathcal{PN}_*, Q, e, t)$ answers "yes" if and only if **D-WMaxSAT**$(\Psi_{\mathcal{PN}_*,Q,e,t}, X + \log_2\ t + M * (Z + |E|))$ answers "yes" where π_***-D-MAP** is given by Definition 6.3 and **D-WMaxSAT** is given by Definition 5.4.

Proof. Let us first recall the parameters of the WMaxSAT decision problem, **D-WMaxSAT**$(\Psi_{\mathcal{PN}_*,Q,e,t}, k)$. Namely,

- $\Psi_{\mathcal{PN}_*,Q,e,t}$ is the weighted CNF formula given by Definition 6.6.
- k is the threshold for the problem and it is given by:
$$k = X + \log_2 t + M * ((\sum \Pi_{\mathcal{PN}_*}(x_i|u_i) = 0) + 1) \quad (11)$$
where M is defined in Definition 6.6. The value of X is defined by the sum of weights in Ψ_R: $X = \sum\{\alpha_i : (\neg x_i \vee \neg u_{ij}, \alpha_i) \in \Psi_R\}$.

Recall that π_***-D-MAP** decision problem is: Given an instantiation e of evidence variables, is there an instantiation q of query variables Q such that $\Pi(q, e) \geq t$?

Let us now show that the two decision problems π_***-D-MAP**$(\mathcal{PN}_*, Q, e, t)$ and **D-WMaxSAT**$(\Psi_{\mathcal{PN}_*,Q,e,t}, X + \log_2\ t + M * (Z + |E|))$ are equivalent. Let the query associated to **D-WMaxSAT** be: Does **D-WMaxSAT**$(\Psi_{\mathcal{PN}_*,Q,e,t}, X + \log_2 t + M * (Z + 1))$ answer "yes"? More precisely, is there an instantiation of all variables that satisfies a subset of clauses in $\Psi_{\mathcal{PN}_*,Q,e,t}$ having the sum of the degrees of the satisfied clauses greater or equal to k?

For the sake of clarity, in this proof, we simply write Ψ instead of $\Psi_{\mathcal{PN}_*,Q,e,t}$.
$\star$ Assume that **D-WMaxSAT**(Ψ, k) answers "yes". This means that there exists a subset $A \subseteq \Psi$ such that:

- $\{(\phi_i, \alpha_i) \in A\}$ is consistent and
- $\sum_{(\phi_i,\alpha_i)\in A} \alpha_i \geq k$

Note that we can state that $\{(e_k, M) : k = 1, ..., l\}$ is included in A. Indeed, if some (ϕ_i, M) of Ψ is not in A then $(\sum_{(\phi_i,\alpha_i)\in A} \alpha_i)$ cannot be greater than $M * (Z + |E|)$. Let us denote by $A^* = A \setminus \{(\phi_i, M) : (\phi_i, M) \in A\}$ then we can also state that:

- $\{(\phi_i, \alpha_i) \in A^*\}$ is consistent,
- $\sum_{(\phi_i,\alpha_i)\in A^*} \alpha_i \geq X + \log_2\ t$

Let ω be a model of $\{\phi_i : (\phi_i, \alpha_i) \in A\}$ and $\{\phi_i : (\phi_i, \alpha_i) \in A^*\}$. Since $X = \sum\{\alpha_i : (\phi_i, \alpha_i) \in \Psi$ and $\alpha_i \neq M\}$. Then the latter equation implies that:

$$\sum\{\alpha_i : (\phi_i, \alpha_i) \notin A^*\} \leq -\log_2\ t$$

This can be rewritten as:

$$\sum\{\alpha_i : (\phi_i, \alpha_i) \in \Psi \setminus A,\ \omega \not\models \phi_i\} \leq -\log_2\ t$$

It is sufficient now to consider the following simplified inequalities to have the wanted result.

$$\begin{array}{ll}
\sum\{\alpha_i : (\phi_i, \alpha_i) \in \Psi \setminus A,\ \omega \nvDash \phi_i\} & \leq -\log_2\ t \\
-\sum\{\log_2 2^{-\alpha_i} : (\phi_i, \alpha_i) \in \Psi \setminus A,\ \omega \nvDash \phi_i\} & \leq -\log_2\ t \\
-\log_2(*\{2^{-\alpha_i} : (\phi_i, \alpha_i) \in \Psi \setminus A,\ \omega \nvDash \phi_i\}) & \leq -\log_2\ t \\
-\log_2(*\{2^{-\alpha_i} : \omega \nvDash \neg x_i \vee \neg u_{ij}\}) & \leq -\log_2\ t \\
-\log_2(*\{2^{-\alpha_i} : \omega \vDash x_i \wedge u_{ij}\}) & \leq -\log_2\ t \\
-\log_2\ \pi_{\mathcal{PN}_*}(\omega) & \leq -\log_2\ t \\
\pi_{\mathcal{PN}_*}(\omega) & \geq t
\end{array}$$

with $\omega \vDash e$. Hence the answer to π_***-D-MAP**$(\mathcal{PN}_*, Q, e, t)$ is also "yes" by taking q such that $\omega \models q$.

$\star$ Assume that **D-WMaxSAT**$(\Psi_{\mathcal{PN}_*,Q,e,t}, k)$ answers "no". Then, for all consistent subset of clauses A that include Ψ_0 and Ψ_e we have

$$\sum\{\alpha_i : (\phi_i, \alpha_i) \in A\} < k.$$

Let us consider such a subset A_*. Let ω be a model of A^*, then following the same previous steps we have:

$$\begin{array}{ll}
\sum\{\alpha_i : (\phi_i, \alpha_i) \in \Psi \setminus A_*\ s.t\ \omega \nvDash \phi_i\} & > -\log_2 t \\
-\log_2(*\{2^{-\alpha_i} : \omega \nvDash \neg x_i \vee \neg u_{ij}\}) & > -\log_2\ t \\
-\log_2(*\{2^{-\alpha_i} : \omega \vDash x_i \wedge u_{ij}\}) & > -\log_2\ t \\
-\log_2\ \pi_{\mathcal{PN}_*}(\omega) & > -\log_2\ t \\
\pi_{\mathcal{PN}_*}(\omega) & < t
\end{array}$$

with $\omega \vDash e$. Hence the answer to π_***-D-MAP**$(\mathcal{PN}_*, Q, e, t)$ is also "no". □

The next example illustrates Theorem 6.3.

Example 6.6. Let us follow Example 6.5. Let $Q = \{B\}$ and $E = \{C\}$ be the query variables and evidence variables respectively. Assume the evidence $e = \neg c$. Let $\Psi_{\mathcal{PN}_*,Q,e,t}$ be the weighted CNF formula associated to $\mathcal{PN}_*$ given by Definition 6.6. The *MAP* query over $\mathcal{PN}_*$ is:

> Is there an instantiation q of the variables Q such that $\Pi_{\mathcal{PN}_*}(q, e) \geq 2^{-2}$?

Hence, the corresponding problem **D-WMaxSAT**$(\Psi_{\mathcal{PN}_*,Q,e,t}, k)$ is given by:

> Is there an instantiation of the variables such that the sum of the degrees of the satisfied clauses is greater or equal to k?

Let us set the values of the variables X, M and Z: $X = 21$, $M = 30$, and $Z = 2$. Then, $k = X + \log_2 t + 30 * (Z + 1) = 109$. Given this configuration, **D-WMaxSAT**$(\Psi_{\mathcal{PN}_{*,\{B\},\neg c,2^{-2}}}, 109)$ answers "yes". Indeed, it is enough to consider A such that

$$A = \left\{ \begin{array}{c} (a, 4), \\ (\neg b, 8), \\ (\neg c \vee \neg a \vee \neg b, 7), \\ (\neg c \vee a \vee b, 30), \\ (\neg c \vee a \vee \neg b, 30), \\ (\neg c, 30) \end{array} \right\}$$

The sum of the weights in A is equal to 109. A model of formulas in A can be $a\neg b\neg c$ for which using the product-based chain rule has a possibility degree of $\Pi_{\mathcal{PN}_*}(a\neg bc) = 2^{-2}$. Hence, π_***-D-MAP**$(\mathcal{PN}_*, \{B\}, \neg c, 2^{-2})$ answers "yes" as well.

In this section, we have shown that the complexity of *MAP* inference in possibilistic networks is *NP*-complete. We have also provided the transformations that encode a possibilistic network into a satisfiability problem in order to use the power of SAT solvers. These results are significant as it overrides the complexity for the same queries in Bayesian networks. In the next section, we provide, following the same hypothesis the proof of hardness and completeness for *MPE* query in possibilistic networks.

7. Analysis of *MPE* querying a possibilistic network

This section briefly focuses on *MPE* query in possibilistic networks where we will follow the same steps as for showing the computational complexity of *MAP* querying.

7.1. *From 3SAT to MPE querying over B&B possibilistic networks*

In the previous section, we have shown that *MAP* querying a min-based B&B possibilistic network and *MAP* querying a product-based B&B possibilistic network give the same result. This results is also valid for a *MPE* query as shown below.

Proposition 7.1. *Let e be an instantiation of evidence variables. Let $\mathcal{PN}_{B\&B_m}$ and $\mathcal{PN}_{B\&B_*}$ be two B&B possibilistic networks such that $\forall X_i$, $\forall \mu$ an instance of parents of X_i, $\pi_{\mathcal{PN}_{B\&B_m}}(X_i|\mu) = \pi_{\mathcal{PN}_{B\&B_*}}(X_i|\mu)$. Then the answer to π_m-**D-MPE**$(\mathcal{PN}_{B\&B_m}, e, 1)$ is "yes" if and only if the answer to π_*-**D-MPE**$(\mathcal{PN}_{B\&B_*}, e, 1)$ is "yes".*

Proof. Assume that π_m**-D-MPE**$(\mathcal{PN}_{B\&B_m}, e, 1)$ is "yes". This means that there exists an interpretation ω such that $\pi_m(\omega) = 1$ and for all conditionals, involved in the computation of $\pi_m(\omega)$, $\pi_m(x_i|par(x_i)) = 1$. By definition of $\mathcal{PN}_{B\&B_*}$, we have $\pi_*(x_i|par(x_i)) = 1$ and using the product-based chain rule, we obtain that $\pi_*(\omega) = 1$ so π_***-D-MPE**$(\mathcal{PN}_{B\&B_*}, e, 1)$ is "yes". The same reasoning can be used to prove the 'only if' condition. □

7.1.1. *Reduction from 3SAT problem to **B&B-D-MPE** problem*

In the previous sections, we gave the transformation definition of a 3CNF to a B&B possibilistic network in the context of a *MAP* query. In the following, we provide the same definition for a *MPE* query. We first formally define the **B&B-D-MPE** problem.

Definition 7.1. By **B&B-D-MPE**($\mathcal{PN}_{B\&B}, e$) we denote the decision problem associated with *MPE* querying a Boolean and Binary possibilistic network that we define by:
Input: The input of this decision problem is composed of two elements :

- $\mathcal{PN}_{B\&B_m}$: a B&B possibilistic network over $V = \{X_1, ..., X_n\}$ (*min*-based or product-based)
- e (evidence): an instantiation of a set of observation variables E

Question: Is there an instantiation x of variables X such that $\Pi_{\mathcal{PN}_{B\&B}}(x, e) = 1$?

As for *MAP* inference, we build a B&B possibilistic network from a 3CNF. Definition 6.2 given for the *MAP* inference in the previous section can be reused to transform the 3CNF into a B&B possibilistic network. Indeed, the difference between *MAP* and *MPE* inference in B&B possibilistic network lies in the presence of a subset of query variables. The set of variables Q is not used in the definition of the transformation.

Theorem 7.1 provides the reduction from the decision problem **D-3SAT**(Ψ) into **B&B-D-MPE**($\mathcal{PN}_\Psi, e$) where the input e is let to e_Ψ. More formally:

Theorem 7.1. Let Ψ be a 3CNF formula. Let $\mathcal{PN}_\Psi$ be the B&B possibilistic network given by Definition 6.2. Let $V_{\mathcal{PN}_\Psi}$ be the set of variables in $\mathcal{PN}_\Psi$, namely $\{X_1, ..., X_n\} \cup \{C_1, ..., C_m\} \cup \{E_\Psi\}$. Then, **D-3SAT**($\Psi$) answer is "yes" if and only if the **B&B-D-MPE**($\mathcal{PN}_\Psi, e_\Psi$) answers "yes" where **D-3SAT** is given in Definition 5.3 and **B&B-D-MPE** is given by Definition 7.1.

The proof of Theorem 7.1 is the same as the proof of Theorem 6.1. It is even shorter as we don't have to restrict the model instantiation to the variables in Q.

Note that it is clear that *MAP* is a generalization of *MPE* where, in *MPE*, Q is set to the remaining variables not used in E. This explains why it is easier in this second part to prove that *MPE* queries in possibilistic networks are *NP*-complete.

7.2. *From MPE querying a* min*-based possibilistic network to SAT*

The decision problem associated with a *MPE* query in min-based possibilistic networks, denoted π_m-**D-MPE** is defined by:

Definition 7.2. We denote π_m**-D-MPE**$(\mathcal{PN}_m, e, t)$ the decision problem associated with *MPE* querying a min-based possibilistic network. It is defined by:
Input: The input of this decision problem is composed of three elements :

- $\mathcal{PN}_m$: a min-based possibilistic network
- e (evidence): an instantiation of a set of variables E
- t: a real number in $(0, 1]$.

Question: Is there an instantiation x of the variables X such that $\Pi_{\mathcal{PN}_m}(x, e) \geq t$?

The definition of $\Psi_{\mathcal{PN}_m,e,t}$, the CNF formula associated to a min-based possibilistic network for the *MPE* query with evidence e and threshold t is given by $\Psi'_{\mathcal{PN}_m,\emptyset,e,t}$ where Ψ' is given by definition 6.4.

The following theorem states that π_m**-D-MPE** can be reduced to **D-SAT**.

Theorem 7.2. Let $\mathcal{PN}_m$ be a min-based possibilistic network, e be an instantiation of evidence variables E and t be a real number in $(0, 1]$. Let $\Psi_{PN_m,e,t}$ be the CNF formula given by Definition 6.4 with $Q = \emptyset$. Then, π_m**-D-MPE**$(\mathcal{PN}_m, e, t)$ says "yes" if and only if **D-SAT**$(\Psi_{\mathcal{PN}_m,e,t})$ says "yes" where π_m**-D-MPE** is given by Definition 7.2 and **D-SAT** is given by Definition 5.2.

Proof. We need to prove that when $\Psi_{\mathcal{PN}_m,e,t}$ is satisfiable then $\Pi_{\mathcal{PN}_m}(x, e) \geq t$ and that when $\Psi_{\mathcal{PN}_m,e,t}$ is unsatisfiable then $\Pi_{\mathcal{PN}_m}(x, e) < t$ for all assignments of all variables compatible with e.

- Assume that $\Psi_{\mathcal{PN}_m,e,t}$ is satisfiable. This means that there exists an instantiation of all variables, denoted by ω^*, that satisfies all clauses of $\Psi_{\mathcal{PN}_m,e,t}$ including $e = e_1, ..., e_l$. Then we have $\pi_{\mathcal{PN}_m}(x_i|u_{ij}) < t$ by construction of $\Psi_{\mathcal{PN}_m,e,t}$. So if ω^* satisfies all clauses in $\Psi_{\mathcal{PN}_m,e,t}$ then ω^* falsifies each of the formulas in $\{(x_i \wedge u_{ij}) : (\neg x_i \vee \neg u_{ij}) \in \Psi_{\mathcal{PN}_m,e,t}\}$. Thus, all conditionals $\pi_{\mathcal{PN}_m}(x_i|u_{ij})$ applied in chain rule to compute $\pi_{\mathcal{PN}_m}(\omega^*)$ have a possibility degree greater or equal to t. Therefore, $\pi_{\mathcal{PN}_m}(\omega^*) \geq t$. Hence the answer to π_m**-D-MPE**$(\mathcal{PN}_m, e, t)$ is also "yes".
- Assume that $\Psi_{\mathcal{PN}_m,e,t}$ is unsatisfiable. Then for all instantiation of variables ω such that $\omega \models e(= e_1 \wedge .. \wedge e_l)$, there exists at least a clause $C_i = \neg x_i \vee \neg u_{ij}$ that is falsified by ω (and hence $\omega \models x_i \wedge u_{ij}$). Again by construction of $\Psi_{\mathcal{PN}_m,e,t}$, we have $\pi_{\mathcal{PN}_m}(x_i|u_{ij}) < t$, so using the min-based chain rule we have $\forall \omega \models e$, $\pi_{\mathcal{PN}_m}(\omega) < t$. Hence π_m**-D-MPE**$(\mathcal{PN}_m, e, t)$ is also "no". □

7.3. *From MPE querying a product-based possibilistic network to WMaxSAT*

The decision problem associated with a *MPE* query in product-based possibilistic networks, denoted π_***-D-MPE** is defined by:

Definition 7.3. We denote π_***-D-MPE**$(\mathcal{PN}_*, e, t)$ the decision problem associated with *MPE* querying a product-based possibilistic network. It is defined by:

Input: The input of this decision problem is composed of three elements :

- $\mathcal{PN}_*$: a product-based possibilistic network
- e (evidence): an instantiation of a set of variables E
- t: a real number in $(0, 1]$.

Question: Is there an instantiation x of the variables X such that $\Pi_{\mathcal{PN}_*}(x, e) \geq t$?

The definition of $\Psi_{\mathcal{PN}_*,e,t}$, the CNF formula associated to a product-based possibilistic network for the *MPE* query with evidence e and threshold t is given by $\Psi'_{\mathcal{PN}_*,\emptyset,e,t}$ where Ψ' is given by definition 6.6.

Theorem 7.3 provides the reduction from the decision problem π_***-D-MPE**$(\mathcal{PN}_*, e, t)$ into **D-WMaxSAT** $(\Psi_{\mathcal{PN}_*,e,t}, k)$. We denote (in the same way as for the *MAP* analysis) by Z the number of possibility degrees, $\pi_{\mathcal{PN}_*}(x_i|u_{ij})$ in $\mathcal{PN}_*$ that are equal to 0.

The input k is let to $X + log_2\ t + M * (Z + |E|)$ while $\Psi_{\mathcal{PN}_*,e,t}$ is the weighted CNF formula given associated to $\mathcal{PN}_*$ given by Definition 6.6 where Q is let to the empty set. More formally:

Theorem 7.3. Let $\mathcal{PN}_*$ be a product-based possibilistic network. Let e be an instantiation of variables E and t be a threshold. Let $\Psi_{\mathcal{PN}_*,e,t}$ be the CNF formula given by Definition 6.6. Then, π_***-D-MPE**$(\mathcal{PN}_*, e, t)$ answers "yes" if and only if **D-WMaxSAT**$(\Psi_{\mathcal{PN}_*,e,t}, X + log_2\ t + M * (Z + |E|))$ answers "yes" where π_***-D-MPE** is given by Definition 6.3 and **D-WMaxSAT** is given by Definition 5.4.

The proof follows the same reasoning as the proof of Theorem 6.3.

To summarise Theorems 7.1, 7.2 and 7.3 show that the decision problem associated with *MPE* inference is *NP*-complete for both min-based and product-based possibilistic networks.

8. Conclusions

In the motivations section, we stressed out the fact that inference in probabilistic graphical models is a hard task in the general case. For instance, answering *MAP* queries in Bayesian networks is NP^{PP}-complete [12, 29]. In this paper, we provide complexity results for possibilistic networks. More precisely, *MAP* inference queries are shown to be *NP*-complete. Moreover, these results are valid in both min-based and product-based possibilistic networks. This paper also shows that the complexity of *MPE* inference is also *NP*-complete. These results suggest that possibilistic networks offer nice features and interesting advantages for reasoning with uncertain information.

As future work lines, we will first investigate the study of computational complexity in interval-based possibilistic networks which extend standard possibilistic networks to allow assessing uncertainty by intervals of degrees instead of point-wise degrees. We think that our results on *MAP* and *MPE* queries will still hold in the interval-based setting. Indeed, in interval-based possibilistic logic, the complexity of conditioning is the same as the complexity of conditioning a standard possibilistic knowledge base. We also believe that the nice complexity results of inference in possibilistic networks shown in this paper can benefit for inference in credal networks by approximating inference in this latter through inference in possibilistic networks by means of imprecise probability-possibility transformations [30, 31].

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